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Analysis of a stress singularity in a non-linear Flamant problem for certain models of a material☆

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ABSTRACT

A generalized plane problem in the non-linear theory of elasticity is considered for a half-plane loaded on the boundary with a concentrated external force (the non-linear Flamant problem). The properties of the material of the half-plane are described by different (known) models, and each model of the non-linearly elastic material generates its own specific boundary-value problem. Analytical solutions of the problems are obtained for two models of an incompressible material: the neo-Hookean model and the Bartenev–Khazanovich model, and a model of a compressible semi-linear (harmonic) material. The dependence of the stress state as a whole on the adopted model of the material and the effect of the model of the material on the form of the stress singularity in the neighbourhood of a pole are investigated.

Solutions of the classical Boussinesq and Flamant[1,2] problems, obtained using the equations of the linear theory of elasticity, have a number of deficiencies. In the linear Flamant problem, only the radial stress is non zero, it has a singularity of the simple pole type and the solution is independent of the mechanical properties of the material. The displacements have a logarithmic singularity at the pole and suffer a discontinuity at the boundary of the half-plane.

The question of the admissibility of the use of the equations of the linear theory of elasticity to solve singular boundary-value problems for domains with cuts, concentrated loads, etc. is of current interest. In problems of this class, the deformations are not only not small, as linear theory assumes, but they become infinite in the neighbourhood of a singular point.

In recent years, papers have appeared where these problems were solved using the equations of the non-linear theory of elasticity. The non-linear Boussinesq problem has been considered[3–13] for several models of an isotropic and transversely isotropic material. However, only asymptotic solutions of the problems have been constructed for the neighbourhood of the point of application of the force, during which assumptions were used which restrict the generality, in particular, concerning the form of the singularity of the solution in the neighbourhood of the pole. A stress singularity exponent was obtained in some papers which depends on the elastic moduli of the material. It has been noted that not all of the models of a material which have been considered ensure the finiteness of the displacements at the point of application of the force. In some cases, where the displacements are finite, the non-linear problems have been solved numerically.

We have already considered the non-linear Flamant problem in Refs 14 and 15 for the case of a model of an incompressible neo-Hookean material where, initially, only the stress state was found, in particular, in the neighbourhood of the pole[15] and the displacement field was then analysed[15]. It was shown that, in the non-linear formulation of the Flamant problem, all of the well-known contradictions of the displacements of the linear solution, that is, the displacements do not have a logarithmic singularity and are continuous at the pole, are eliminated. A solution of the non-linear problem for an incompressible Bartenev-Khazanovich material has been obtained[16]. It is known that this model sometimes leads to results which contradict physical concepts, which also appeared in the solution of the Flamant problem, and this casts doubt on the physical authenticity of the stress state in the neighbourhood of the pole.

The Flamant problem for a model of a compressible semi-linear material is solved for the first time below. Unlike existing results[3–13], which are local asymptotic solutions for the neighbourhood of the point of application of a concentrated force, global solutions of the equations of the plane non-linear theory of elasticity are found for the whole domain and only then are the asymptotic expansions in the neighbourhood of the pole and at infinity constructed.

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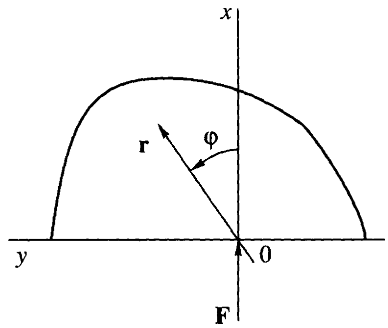

Fig. 1.

1. Formulation of the boundary-value problem

In the cylindrical coordinates of the reference configuration (r, φ, z), $r \in [0,\infty)$, $|\varphi| \le \pi/2$ corresponds to the half-plane being considered (Fig. 1). We will assume that the basis vectors $\mathbf{e}_i (i = 1,2,3)$ of the system of coordinates coincide with the principal axes of the Cauchy strain tensor. As a result of deformation the vectors $\mathbf{e}_i$ become the vectors $\boldsymbol{e}'_i$ which also form an orthonormal basis. This assumption is satisfied in linear Flamant and Mitchell problems[17] which provide specific grounds for its acceptance. The specific grounds for the correctness of this hypothesis have been presented in Refs 14 and 15.

The Cauchy strain tensor is diagonal in the basis $\mathbf{e}_i$: $\boldsymbol{C} = \lambda_\alpha^2 \boldsymbol{e}_\alpha \boldsymbol{e}_\alpha$. The gradient of the deformation $\mathbf{G}$ and the conditional (nominal) stress tensor $\mathbf{S}$ in the mixed vector basis will also be diagonal[18]:

$$\mathbf{G} = \lambda_\alpha \mathbf{e}'_\alpha \mathbf{e}_\alpha, \quad \mathbf{S} = s_\alpha \mathbf{e}_\alpha \mathbf{e}'_\alpha, \tag{1.1}$$

where λ_i are the principal expansion ratios, and the expansion ratio in the axial direction $\lambda_3 = \text{const}$. For the plane problem, the parameter $\lambda_3 = 1$, and, for the generalized plane problem, $\lambda_3 \neq 1$ is a specified number or quantity which is determined from the known axial force.

The boundary conditions of the problem have the form

$$\mathbf{s}_1, \mathbf{s}_2 \to \mathbf{0} \text{ when } r \to \infty, \quad \mathbf{s}_2 = \mathbf{0} \text{ when } \varphi = \pm\pi/2 \tag{1.2}$$

where $\mathbf{s}_i = \mathbf{e}_i \cdot \mathbf{S}$ are the conventional stress vectors. The resultant of the stresses $\mathbf{s}_1$ in an arc of a circle of radius $r = \text{const}$ must balance the external force $\mathbf{F}$:

$$\int_{-\pi/2}^{+\pi/2} \mathbf{s}_1 r d\varphi = -\mathbf{F} \tag{1.3}$$

When $r \to 0$, the stresses increase such that the force $\mathbf{F} = \text{const}$.

2. A neo-Hookean material

The constitutive relations for a neo-Hookean type material for the conventional stress tensor $\boldsymbol{S} = s_\alpha \boldsymbol{e}_\alpha \boldsymbol{e}'_\alpha$, the true Cauchy stress tensor $\boldsymbol{T} = t_\alpha \boldsymbol{e}'_\alpha \boldsymbol{e}'_\alpha$ and the Piola-Kirchhoff stress tensor $\boldsymbol{\Sigma} = \sigma_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha$ have the form[18]

$$\mathbf{S} = \mu \mathbf{G}^T + qJ\mathbf{G}^{-1}, \quad J\mathbf{T} = \mu \mathbf{B} + qJ\mathbf{I}, \quad \boldsymbol{\Sigma} = \mu \mathbf{I} + qJ\mathbf{C}^{-1} \tag{2.1}$$

where μ is the shear modulus, $J = \lambda_1\lambda_2\lambda_3$ is volume change ratio, $\mathbf{C} = \mathbf{G}^T \cdot \mathbf{G}$ is the first (right) Cauchy strain tensor, $\mathbf{B} = \mathbf{G} \cdot \mathbf{G}^T$ is the second (left) Cauchy strain tensor and $\boldsymbol{\Lambda} = \lambda_\alpha \mathbf{e}_\alpha \mathbf{e}_\alpha$ is the expansion ratio tensor. The parameter q is an unknown function of the variables r and φ in the case of an incompressible material.

The stresses are connected by the relations

$$Jt_i = \lambda_i s_i = \lambda_i^2 \sigma_i = \mu \lambda_i^2 + qJ, \quad i = 1, 2, 3 \tag{2.2}$$

It follows from the stress conditions (1.2) and formulae (2.2) that

$$q \to -\mu \lambda_3^{-1}, \quad \lambda_i^2 \to J\lambda_3^{-1}, \quad i = 1, 2, \quad r \to \infty$$

We will write the equilibrium equations of the reference configuration in projections onto the axes of the deformed basis $\boldsymbol{e}'_i$ (there are no bulk forces)

$$\lambda_1 (rs_1)'_r - (r\lambda_2)'_r s_2 = 0, \quad \lambda_2 (s_2)'_\varphi - (\lambda_1)'_\varphi s_1 = 0 \tag{2.3}$$

In the case of the law (2.1), we obtain from Eqs (2.3)

$$\mu r(\lambda_1^2 - \lambda_2^2) = 2h(\varphi) + rg'(r) - g(r), \quad 2rq = 2h(\varphi) - [rg'(r) + g(r)] \tag{2.4}$$

where $g(r)$ and $h(\varphi)$ are unknown integration functions.

We find the expansion ratios and the function q from formula (2.4) and the incompressibility condition $J = 1$, and we then obtain the function $g(r) = \mu r\lambda_3^{-1}$ from the boundary condition (1.2). The principal functions which are required are defined by the expressions[14]

$$\mu r\lambda_{1,2}^2 = \pm h + \sqrt{h^2 + \mu^2\lambda_3^{-2}r^2}; \quad rq = h - \mu\lambda_3^{-1}r \tag{2.5}$$

The function $h(\varphi)$ still remains unknown. We find it using the equations for the compatibility of the stresses.

The expansion ratio tensor $\mathbf{\Lambda}$ has already been found in the polar expansion of the gradient of the strain tensor $\mathbf{G} = \mathbf{Q}\cdot\mathbf{\Lambda}$. The general form of the orthogonal tensor for the plane problem in the basis of the reference configuration is[18]

$$\mathbf{Q} = \cos\omega(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2) - \sin\omega(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) + \mathbf{e}_3\mathbf{e}_3$$

where ω is the angle of rotation of the principal axes as a result of deformation. Using the formulae for the differentiation of vectors $\boldsymbol{e}_i' = \mathbf{Q}\cdot\boldsymbol{e}_i$, we obtain the two equations[15] for the function ω

$$\frac{\partial\omega}{\partial r} = -\frac{1}{r\lambda_2}\frac{\partial\lambda_1}{\partial\varphi}, \quad 1 + \frac{\partial\omega}{\partial\varphi} = \frac{1}{\lambda_1}\frac{\partial r\lambda_2}{\partial r}$$

and we write the condition for the compatibility of the equations (it is the same as the condition for the compatibility of the deformations)

$$\frac{\partial}{\partial r}\frac{1}{\lambda_1}\frac{\partial r\lambda_2}{\partial r} + \frac{\partial}{\partial\varphi}\frac{1}{r\lambda_2}\frac{\partial\lambda_1}{\partial\varphi} = 0 \tag{2.6}$$

We substitute the expansion ratios (2.5) into condition (2.6) and take the limit as $r \to \infty$. As a result, we obtain the equation

$$h''_\varphi + h = 0$$

the solution of which, that satisfies condition (1.3), is

$$h = -\frac{1}{\pi}F\lambda_3^{-1/2}\cos\varphi, \quad |\varphi| \le \frac{\pi}{2} \tag{2.7}$$

where F is the magnitude of the external force (per unit thickness of the body). The conventional stresses, the true Cauchy stresses, and the Piola–Kirchhoff stresses are calculated using formulae (2.2) where the expansion ratios λ_1 and λ_2 and the function q are completely determined. In view of relations (2.2), the stresses have different asymptotic forms in the neighbourhood of the pole.

In particular, when $r \to \infty$, we have the expansions

$$\lambda_1^2 = -\frac{\mu\lambda_3^{-2}r}{2h} + O(r^3), \quad \lambda_2^2 = -\frac{2h}{\mu r} + \lambda_1^2 \tag{2.8}$$

$$t_1 = \frac{h}{r} - \mu\lambda_3^{-1} + O(r), \quad t_2 = -\frac{h}{r} - \mu\lambda_3^{-1} + O(r) \tag{2.9}$$

$$s_1 = \sqrt{\frac{2|h|}{\mu r}}\left[\frac{h\lambda_3}{r} - \mu + O(r)\right], \quad s_2 = \sqrt{\frac{\mu r}{2|h|}}\left[-\frac{h}{r} - \mu\lambda_3^{-1} + O(r)\right] \tag{2.10}$$

$$\sigma_1 = -\frac{2h^2\lambda_3^{-2}}{\mu r^2} + \frac{2h\lambda_3}{r} + \mu + O(r), \quad \sigma_2 = \mu + O(r) \tag{2.11}$$

The asymptotic expansions for $r \to \infty$ are

$$\lambda_1^2 = \lambda_3^{-1} + \frac{h}{\mu r} + O(r^{-2}), \quad \lambda_2^2 = -\frac{2h}{\mu r} + \lambda_1^2$$

$$t_1 = \frac{2h}{r} + \frac{h^2\lambda_3}{2\mu r^2} + O(r^{-4}), \quad t_2 = \frac{h^2\lambda_3}{2\mu r^2} + O(r^{-4})$$

$$s_1 = 2\frac{h\lambda_3^{1/2}}{r}\left[1 - \frac{h\lambda_3}{4\mu r}\right] + O(r^{-4}), \quad s_2 = \frac{h^2\lambda_3^{3/2}}{2\mu r^2}\left[1 + \frac{h\lambda_3}{2\mu r}\right] + O(r^{-4})$$

$$\sigma_1 = 2\frac{h\lambda_3}{r} - \frac{3}{2}\frac{h^2\lambda_3^2}{\mu r^2} + O(r^{-3}), \quad \sigma_2 = \frac{h^2\lambda_3^2}{2\mu r^2} + O(r^{-3})$$

A special investigation of the asymptotic forms of the solution is required in the neighbourhood of the boundaries $\varphi = \pm\pi/2$. Boundaries the stresses t_i, s_i, σ_i ($i = 1,2$) vanish, and the other quantities take the values $\lambda_1^2 = \lambda_2^2 = \lambda_3^{-1}$, $s_3 = \mu(\lambda_3 - \lambda_3^{-2})$.

Together with the radial stresses, the solution of the non-linear Flamant problem also contains the peripheral stresses, which are equal to zero in the linear problem, and the true peripheral Cauchy stresses, which are tensile stresses, have the same singularity in the neighbourhood of the pole as the radial stresses.

3. A Bartenev–Khazanovich material

An incompressible elastomer is considered, the properties of which are specified by the Bartenev–Khazanovich potential

$$\Phi = 2\mu(\mathrm{tr}\boldsymbol{\Lambda} - 3)$$

The constitutive relations for the stress tensors have the form[16]

$$\mathbf{S} = 2\mu\mathbf{Q}^T + qJ\mathbf{G}^{-1}, \quad J\mathbf{T} = 2\mu\mathbf{Q}\cdot\boldsymbol{\Lambda}\cdot\mathbf{Q}^T + qJ\mathbf{I}, \quad \boldsymbol{\Sigma} = 2\mu\boldsymbol{\Lambda}^{-1} + qJ\boldsymbol{\Lambda}^{-2}$$

For the stresses, we have the relations

$$Jt_i = \lambda_i s_i = \lambda_i^2\sigma_i = 2\mu\lambda_i + qJ, \quad i = 1, 2, 3 \tag{3.1}$$

where q is an unknown function of the variables r and φ. From the conditions that $s_i \to 0$ when $r \to \infty$, it follows that

$$\lambda_i \to \lambda_3^{-1/2}, \quad Jq \to -2\mu\lambda_3^{-1/2}$$

Integrating equilibrium equations (2.3), where the stresses are given by formulae (3.1), we obtain

$$2\mu\lambda_1 = q + g'(r), \quad 2\mu r\lambda_2 = h(\varphi) + rq + g(r) \tag{3.2}$$

Using relations (3.2) and the incompressibility condition, we express the expansion ratios and the function q in terms of the unknown integration functions $g(r)$ and $h(\varphi)$. The function $g(r)$ is found from the boundary conditions $s_2 = 0$ in the rays $\varphi = \pm\pi/2$: $g(r) = 4\pi r\lambda_3^{-1/2}$.

The expansion ratios and the function q have the form[16]

$$4\mu r\lambda_{1,2} = \mp h + \sqrt{h^2 + 16\mu^2\lambda_3^{-1}r^2}, \quad 2rq = -h + \sqrt{h^2 + 16\mu^2\lambda_3^{-1}r^2} - 8\mu\lambda_3^{-1/2}r$$

The function $h(\varphi)$ is found from the compatibility condition (2.6) using the previous method

$$h(\varphi) = \frac{2}{\pi}F\sqrt{\lambda_3}\cos\varphi, \quad |\varphi| \le \frac{\pi}{2}$$

We now construct the asymptotic expansion of the expansion ratios and the stresses in the neighbourhood of the pole and at infinity[16]. When $r \to \infty$, the asymptotic expansions are

$$\lambda_1 = \frac{2\mu r}{h\lambda_3}\left(1 - \frac{4\mu^2\lambda_3^{-1}r^2}{h^2}\right) + O(r^5), \quad \lambda_2 = \frac{h}{2\mu r} + \lambda_1; \quad q = -4\mu\lambda_3^{-1/2} + 2\mu\lambda_1 \tag{3.3}$$

$$t_1 = -4\mu\lambda_3^{-1/2} + 4\mu\lambda_1, \quad t_2 = \frac{h}{r} + t_1 \tag{3.4}$$

$$s_1 = -\frac{2h\lambda_3^{1/2}}{r} + 4\mu\frac{8\mu^2\lambda_3^{-1/2}r}{h} + O(r^3), \quad s_2 = 2\mu - \frac{8\mu^2\lambda_3^{-1/2}r}{h} + O(r^2) \tag{3.5}$$

$$\sigma_1 = -\frac{h^2\lambda_3^{3/2}}{\mu r^2} + \frac{2h\lambda_3}{r} - 4\mu\lambda_3^{1/2} + O(r), \quad \sigma_2 = \frac{4\mu^2 r}{h} - \frac{16\mu^3\lambda_3^{-1/2}r^2}{h^2} + O(r^3) \tag{3.6}$$

The true radial stresses do not have a singularity at the pole. A special investigation of the asymptotic forms is required in the neighbourhood of the boundary of the domain $\varphi = \pm\pi/2$. On the boundary

$$\lambda_i = \lambda_3^{-1/2}, \quad q = -2\mu\lambda_3^{-1/2}, \quad t_i = s_i = 0, \quad i = 1, 2$$

When $r \to \infty$, the asymptotic expansions are

$$\lambda_1 = \lambda_3^{-1/2} - \frac{h}{4\mu r} + \frac{h^2\lambda_3^{1/2}}{32\mu^2 r^2} + O(r^{-4}), \quad \lambda_2 = \frac{h}{2\mu r} + \lambda_1;$$

$$q = -2\mu\lambda_3^{-1/2} - \frac{h}{2r} + \frac{h^2\lambda_3^{1/2}}{16\mu r^2} + O(r^{-4})$$

$$t_1 = -\frac{h}{r} + \frac{h^2\lambda_3^{1/2}}{8\mu r^2} + O(r^{-4}), \quad t_2 = \frac{h^2\lambda_3^{1/2}}{8\mu r^2} + O(r^{-4})$$

$$s_1 = -\frac{h\lambda_3^{1/2}}{r} - \frac{h^2\lambda_3}{8\mu r^2} + O(r^{-4}), \quad s_2 = \frac{h^2\lambda_3}{8\mu r^2} - \frac{h^3\lambda_3^{3/2}}{32\mu^2 r^3} + O(r^{-4})$$

$$\sigma_1 = -\frac{h\lambda_3}{r} - \frac{3h^2\lambda_3^{3/2}}{8\mu r^2} + O(r^{-3}), \quad \sigma_2 = \frac{h^2\lambda_3^{3/2}}{8\mu r^2} + O(r^{-3})$$

The leading terms in the expansions of the stresses have the same dependence on r as the stresses in the linear Flamant problem[1].

4. A semi-linear material

This material is also called a harmonic material, a John material or a standard first order material. An elastic potential

$$\Phi = \lambda(\mathrm{tr}\mathbf{\Lambda} - 3)^2/2 - 2\mu(\mathrm{tr}\mathbf{\Lambda} - 3) + 2\mu e$$

corresponds to this material, where $e = \mathrm{tr}\mathbf{E}$ is the trace of the Green strain tensor.

The constitutive relations have the form[18]

$$\mathbf{S} = 2\mu\mathbf{G}^T + \kappa\mathbf{Q}^T, \; J\mathbf{T} = 2\mu\mathbf{B} + \kappa\mathbf{Q}\cdot\mathbf{\Lambda}\cdot\mathbf{Q}^T, \; \mathbf{\Sigma} = 2\mu\mathbf{I} + \kappa\mathbf{\Lambda}^{-1}; \; \kappa = \lambda(\mathrm{tr}\mathbf{\Lambda} - 3) - 2\mu \tag{4.1}$$

The general solution of equilibrium equations (2.3) for the stresses

$$Jt_i = \lambda_i s_i = \lambda_i^2\sigma_i = \lambda_i(\kappa + 2\mu\lambda_i)$$

has the form

$$(\lambda + 2\mu)(\lambda_1^2 - \lambda_2^2)/2 - k\lambda_1 = g'(r), \quad kr(\lambda_1 + \lambda_2) = f$$

where

$$k = 3\lambda + 2\mu - \lambda\lambda_3, \quad f = h(\varphi) - rg'(r) - g(r)$$

where $g(r)$ and $h(\varphi)$ are unknown integration functions.

It follows from the relations

$$s_1 + s_2 = 2(\lambda + \mu)(\lambda_1 + \lambda_2) - 2k, \quad s_1 - s_2 = 2\mu(\lambda_1 - \lambda_2)$$

and condition $s_i \to 0$ when $r \to \infty$ that $\lambda_i \to k/[2(\lambda + \mu)]$.

For the multiplicities of the elongations we obtain the expressions

$$\lambda_{1,2} = \frac{(\lambda + 2\mu)f^2/2 + \kappa_{1,2}}{kr[(\lambda + 2\mu)f - k^2 r]}, \quad \kappa_1 = k^2 r^2 g', \quad \kappa_2 = -k^2 r(h - g)$$

From the condition $s_2 = 0$ when $\varphi = \pm\pi/2$, we find

$$g = -Ar, \quad A = k^2/[2(\lambda + \mu)].$$

Then,

$$f = h(\varphi) + k^2 r/(\lambda + \mu)$$

The function $h(\varphi)$ is found in a similar way to the function (2.7)

$$h = -F\frac{\lambda + \mu}{k\pi}\cos\varphi, \quad |\varphi| \le \pm\frac{\pi}{2}$$

We obtain the asymptotic expansions when $r \to 0$

$$\lambda_1 = \frac{h}{2kr} + \frac{A}{k} + \frac{k}{2(\lambda+2\mu)}(1+Br)^{-1}, \quad \lambda_2 = \frac{h}{kr} + \frac{2A}{k} - \lambda_1 \tag{4.2}$$

$$t_1 = \frac{2(\lambda+\mu)}{\lambda_3} + O(r^2), \quad t_2 = \frac{2(\lambda+\mu)}{\lambda_3} - \frac{4k^2}{h\lambda_3}r + O(r^2) \tag{4.3}$$

$$s_1 = \frac{(\lambda+\mu)h}{kr}\left(1 + \frac{Br}{1+Br}\right), \quad s_2 = \frac{(\lambda+\mu)h}{kr}\left(1 - \frac{Br}{1+Br}\right) \tag{4.4}$$

$$\sigma_1 = 2(\lambda+\mu)\left(1 - \frac{2\lambda+3\mu}{\mu}Br\right) + O(r^2), \quad \sigma_2 = 2(\lambda+\mu)(1-Br) + O(r^2)$$

$$B = \frac{\mu k^2}{(\lambda+\mu)(\lambda+2\mu)h} \tag{4.5}$$

Both the true stresses and the Piola–Kirchhoff stresses do not have a singularity at the pole, and the values of the stresses at the pole are independent of the load.

The asymptotic expansions when $r \to \infty$ are

$$\lambda_1 = \frac{A}{k} + \frac{h}{2kr} + \frac{(\lambda+\mu)h}{2\mu kr}\frac{Br}{1+Br}, \quad \lambda_2 = \frac{2A}{k} + \frac{h}{kr} - \lambda_1$$

$$t_1 = \frac{h}{\lambda_3 r}\left(1 - \frac{1}{2Br} + \frac{\lambda h}{2A\mu r} + O(r^{-2})\right), \quad t_2 = \frac{h}{2\lambda_3 Br^2}\left(1 - \frac{1}{Br} - \frac{(\lambda+2\mu)h}{2A\mu r} + O(r^{-2})\right)$$

$$s_1 = \frac{(\lambda+\mu)h}{kr}\left(1 + \frac{Br}{1+Br}\right), \quad s_2 = \frac{2(\lambda+\mu)h}{kr} - s_1$$

$$\sigma_1 = \frac{h}{r}\left(1 - \frac{1}{2Br} - \frac{(\lambda+2\mu)h}{2A\mu r} + O(r^{-2})\right), \quad \sigma_2 = \frac{h}{2Br^2}\left(1 - \frac{1}{Br} + \frac{\lambda h}{2A\mu r} + O(r^{-2})\right)$$

We now find the volume change ratios

$$J = \frac{h^2\lambda_3}{4k^2r^2} + \frac{h\lambda_3}{2(\lambda+\mu)r} + O(1), \quad r \to 0$$

$$J = \frac{k^2\lambda_3}{4(\lambda+\mu)^2} + \frac{h\lambda_3}{2(\lambda+\mu)r} + O(r^{-2}), \quad r \to \infty \tag{4.6}$$

It follows from the first formula that the volume change ratio has a singularity of the order of r^{-2} at the pole. The volume change ratio also has a singularity at the pole in the linear Flamant problem but of another order r^{-1} and with a negative coefficient.

5. Conclusion

We will now sum up some results of the investigation. The stresses, obtained within the limits of the linear Flamant problem, do not contain the elastic parameters of the material, and this is a general property of plane problems of the class being considered.[17] In non-linear Flamant problems, the stress state depends on the model of the material which is adopted. It is especially important that the model of the material turns out to have a fundamental effect on the form of the singularity in the stresses in the neighbourhood of the pole. This conclusion obviously applies to all non-linear singular boundary value problems.

The solutions of all non-linear problems contain both radial and peripheral stresses. The peripheral stresses are tensile in the case of an incompressible material and are not small, and, in particular, both of the true stresses (2.9) and (3.4) have a singularity of the form $1/r$ in the neighbourhood of the pole when $r \to 0$. In the linear Flamant problem, the peripheral stresses are equal to zero.

The expansions of the expansion ratios and stresses when $r \to \infty$ are practically the same in all non-linear problems, at least in the leading terms. Although the peripheral stresses are non-zero, they decrease significantly more rapidly than the radial stresses (as $1/r^2$). This result would be expected since all the solutions pass into the solution of the linear model at infinity.

We now compare the analytical solutions of the non-linear Flamant problem for a neo-Hookean material and a Bartenev–Khazanovich (BKh) material. Both models describe an incompressible material.

In the case of the expansions in the neighbourhood of the pole when $r \to 0$, the differences in the asymptotic forms of the stresses are of a fundamental character. For instance, the true Cauchy stresses for a neo-Hookean material have a singularity of the form $1/r$ at the pole (as in the solution of the linear problem). The true radial stresses (3.4) were obtained in the case of a BKh material, which do not have a singularity at the pole, and, in the case of the peripheral stresses, there is a singularity of the form $1/r$. The conventional radial and peripheral stresses for a neo-Hookean material (2.10) have singularities of the form $1/r^{3/2}$ and $1/r^{1/2}$ respectively, and, in the case of a BKh material, the radial stress (3.5) has a singularity of the form $1/r$ while the peripheral stress does not have a singularity. We note the following in connection with a model of a semi-linear compressible material. The expansions of the true radial and peripheral stresses and the Piola–Kirchhoff stresses (4.3), (4.5) show that the stresses do not have a singularity at the pole. Both of the conventional stresses (4.4) have a singularity of the form $1/r$ at the pole. The volume change ratio (4.6) has a singularity of the form $1/r^2$.

The following conclusion can be drawn from what has been said: while the stresses in the non-linear Flamant problem for a model of a neo-Hookean material do not contradict physical concepts, the stresses for models of a BKh material and a harmonic material do give rise to doubts. This primarily applies to the true radial stresses: the fact that they have no singularity at the pole contradicts physical concepts and is not in accord with the solution of the linear problem.

It is also important to note that, in cases where the stresses do not have a singularity at the pole (the true radial stress and the conventional peripheral stress in the case of a BKh material and both the true stresses and both the Piola–Kirchhoff stresses for a harmonic material), the values of these stresses at the pole are independent of the external load (of the force F), which can be indicative of deficiencies in the given models of a material. It is obvious that it is impossible to compare these stresses with the stresses obtained within the limits of the linear problem.

The stress intensity factors when $r \to 0$ are different in the non-linear and linear problems. The qualitatively different stress distribution in non-linear problems makes it possible to make the strength criteria, which use the linear solutions, more precise.

The displacements in the linear Flamant problem contradict physical concepts in a number of ways, as has been recorded above. An attempt has been made to explain these contradictions by the existence of a plastic domain and other causes, which do not take account of the mathematical model of the problem.[17] However, in the case of the non-linear formulation of the Flamant problem, displacements were obtained which are free from the above-mentioned deficiences.[15] Since there are no plastic deformations in highly elastic elastomers, the reason for the contradictions is the failure to take account of the non-linearity of the deformation in the neighbourhood of the pole.

Different forms of stresses figure in non-linear problems: the true Cauchy stresses, conventional stresses and Piola–Kirchhoff stresses have been considered here. These stresses have different type of singularities a the pole and the question arises as to which of them is to be compared with the stresses obtained within the limits of the linear problem. Here, there is no unique answer. In the case of a neo-Hookean material only the true Cauchy stresses have the same singularity as the linear stresses, and the leading terms in their expansions are independent of the material. It just makes sense to compare the true stresses with the linear stresses. For the two other models of a material, only the conventional stresses have the same type of singularity as the stresses obtained within the limits of the linear problem. The results which have been presented show that it is advisable to use the equations of the non-linear theory of elasticity to solve singular boundary value problems where the stresses and strains are not bounded in the neighbourhoods of singular points.

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